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1973 J. Phys. A: Math. Nucl. Gen. 6 79

(http://iopscience.iop.org/0301-0015/6/1/008)

J. Phys. A: Math., Nucl. Gen., Vol. 6, January 1973. Printed in Great Britain. © 1973

A critical evaluation of theories of direct electron pair production by muons

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MS received 18 August 1972

Abstract. The development of the theory of direct pair production by relativistic muons is traced in an attempt to reconcile the existing theoretical expressions for the cross section. The interrelation of the theoretical treatments and the shortcomings of approximate formulae are examined. Cross sections are expressed in a form suitable for numerical evaluation and a direct comparison of the formulae of various authors is made.

Numerical calculations of energy loss $-\mathrm{d}E/\mathrm{d}x$ are presented and compared with computations of other authors.

1. Introduction

At the present time there is considerable confusion in the field of cosmic ray muon interactions as to the status of the theory of direct electron pair production. This is because many formulae and approximations exist (not all of which are valid) which offer different descriptions of the same phenomenon. The purpose of this work is to trace the logical development of the subject in an attempt to reveal certain shortcomings as well as the areas of general agreement.

Cross sections for pair production have been available for many years. The early work of Bhabha (1935), Nishina *et al* (1935) and Racah (1937) in which the incident particle and field are treated classically, gives an acceptable description of the process only when the energy transferred to the electron–positron pair is small compared with the muon energy. A concise exposition of the theoretical results of Bhabha has been presented by R Davisson in Rossi (1952). The correct form of the cross section in the four regions IS, IN, IIS and IIN as defined by Bhabha is given. It is evident from these formulae that the cross section varies as u^{-1} and u^{-3} at $u \sim 10^{-3}$ and $u \sim 10^{-1}$ respectively (u is defined as the ratio of the transferred energy to the incident energy). Mando and Ronchi (1952) obtained an expression for the mean energy loss $-\mathrm{d}E/\mathrm{d}x$ derived from the cross section of Bhabha. This formula, although not given in the present work, has been extensively used (see, for example, Hayman *et al* (1963) and Kobayakawa (1967)).

Major consideration will be given to those theories in which the process is treated quantum electrodynamically in accordance with the Feynman formalism. The diagrams (a) and (b) of figure 1 respectively correspond to what Bhabha refers to as the 'second' and 'first order' processes; according to Kel'ner (1967) a further set, referred to as the interference diagrams, need not be taken into account in the case when the incident

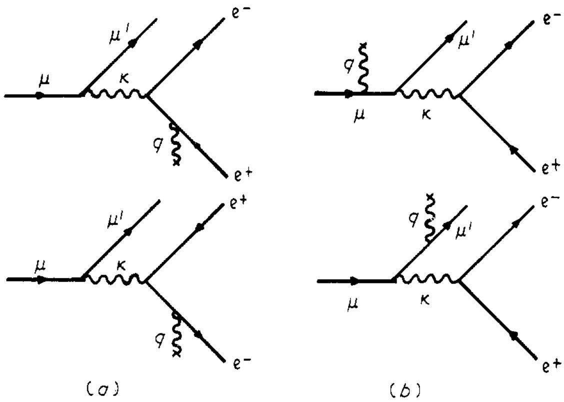

Figure 1. The Feynman–Dyson diagrams for pair production. (*a*) Second order process. (*b*) First order process.

particle is a muon. The cross section comprises the sum of the contributions of the two pairs of diagrams.

$$d\sigma = d\sigma_a + d\sigma_b. \tag{1}$$

The first treatment using the QED formalism was presented by Murota *et al* (1956), (to be referred to as MUT) and gave an indication that the early cross sections were inaccurate. The weakness of the formula presented lay in the appearance of an indeterminate constant α known only to be of the order of unity. The theory of Bhabha contains two such numbers k and k' which were introduced as cut-off parameters in the integration of the differential cross section over the angles of the outgoing particle momenta. This difficulty was overcome by Ternovskii (1960a) and Zapolsky (1962) who arrived at different expressions. In 1967 Kel'ner derived an expression valid for both small and large energy transfers. This was subsequently improved upon by Kel'ner and Kotov (1968) where screening was accounted for more satisfactorily. Finally Kokoulin and Petrukhin (1970) derived a formula based on that of Kel'ner and Kotov which is easier to compute.

2. Differential cross sections

The contributions to the cross section from both diagrams (*a*) and (*b*) of figure 1 are derived subject to the condition that the energies of the participant particles are large compared with their rest energies, that is,

$$\epsilon_+, \epsilon_- \gg m, \qquad E \gg \mu, \qquad E - E' \gg \mu \tag{2}$$

where $E(E-E')$ is the energy of the incident muon before (and after) the interaction, $\epsilon_+(\epsilon_-)$ the energy of the positron (electron) and $m(\mu)$ the rest mass of the electron (muon). These conditions, ensuring that all the participating particles are relativistic, are common to the derivations of all the above-mentioned authors. Cross sections valid for small energy transfers ($E' < m$) have been presented by Bhabha (1935) but will not be treated here.

For the comparison of theory with experiment, a cross section independent of the splitting of the pair energy is most frequently required—the quantity actually measured in the experiment being E'. We therefore express formulae (23) and (42) of MUT in terms of the variables u and v defined as

$$u = \frac{\epsilon_+ + \epsilon_-}{E} = \frac{E'}{E},$$

$$v = \frac{\epsilon_+ - \epsilon_-}{\epsilon_+ + \epsilon_-}, \tag{3}$$

$$\mathrm{d}u\,\mathrm{d}v = \frac{2}{uE^2}\,\mathrm{d}\epsilon_+\,\mathrm{d}\epsilon_-.$$

Transforming equation (23) of MUT by means of (3), we obtain

$$\mathrm{d}^2\sigma_a(u, v) = \frac{1}{3\pi}(Z\alpha' r_0)^2 \frac{\mathrm{d}u\,\mathrm{d}v}{u} L_a\left[\left\{\{(2+v^2)+x(3+v^2)\}\ln\left(1+\frac{1}{x}\right)-(3+v^2)\right\}\{1+(1-u)^2\}\right.$$
$$\left.+\frac{2(1-v^2)(1-u)}{1+x}+u^2\left\{\frac{x}{1+x}+(2+v^2)-x(3+v^2)\ln\left(1+\frac{1}{x}\right)\right\}\right] \tag{4}$$

where

$$x = \left(\frac{\mu}{2m}\right)^2 \frac{u^2(1-v^2)}{1-u},$$

α' is the fine structure constant, r_0 the classical radius of the electron and Z the atomic number. The expression for the screening term L_a is determined by the value of a parameter γ, defined as

$$\gamma = \frac{\delta A'}{mZ^{1/3}} \tag{5}$$

where $\delta = 2m^2(1+x)/Eu(1-v^2)$ is the least momentum transferred to the nucleus and $A'/mZ^{1/3}$ is the effective atomic radius in the Thomas–Fermi model. In the work of MUT, A' is taken as 137, so that

$$\gamma = \frac{2m(1+x)137Z^{-1/3}}{Eu(1-v^2)}. \tag{6}$$

The importance, or otherwise, of screening is determined by the value of γ and is regarded as complete when $\gamma \ll 1$ and absent for $\gamma \gg 1$. For the purpose of computation, the conditions were taken to be $\gamma \leqslant 1$ for complete screening and $\gamma > 1$ for no screening. Different critical values of γ, such as 0·1 and 0·01, were assumed in order to test the dependence of the cross sections on this parameter. The cross sections were found to be substantially independent of the choice of γ for most E and E' of interest, that is, $E > 10$ GeV, $E' > 0{\cdot}01$ GeV.

The expressions for L_a are

$$\begin{aligned} L_a &= \ln\{\alpha 137Z^{-1/3}(1+x)^{1/2}\}, & \gamma \leqslant 1 \\ &= \ln\left(\frac{\alpha}{2}\frac{uE}{me}\frac{(1-v^2)}{(1+x)^{1/2}}\right), & \gamma > 1 \end{aligned} \tag{7}$$

with e = 2·718 . . . .

The cross section for the first order processes, from equation (42) of MUT, is

$$d^2\sigma_b(u, v) = \frac{1}{3\pi}(Z\alpha' r_0)^2\left(\frac{m}{\mu}\right)^2 \frac{du\, dv}{u} L_b\left(\left[\{1+(1-u)^2\}\left\{\left(\frac{3}{2}+\frac{2}{x}\right)\ln(1+x)-2\right\}\right.\right.$$
$$\left.\left.-(1-u)\left\{\left(1+\frac{2}{x}\right)\ln(1+x)-2\right\}\right](1+v^2)\right) \tag{8}$$

where

$$L_b = \ln\left(\frac{2\alpha E(1-u)}{eu\mu(1+1/x)^{1/2}}\right), \qquad \gamma > 1$$
$$= \ln\left\{\alpha 137 Z^{-1/3}\left(1+\frac{1}{x}\right)^{1/2}\right\}, \qquad \gamma \leqslant 1.$$

It is useful to follow Kokoulin and Petrukhin and write

$$d^2\sigma_{a,b} = \frac{1}{3\pi}(Z\alpha' r_0)^2 \frac{du\, dv}{u}\left\{\chi_a + \chi_b\left(\frac{m}{\mu}\right)^2\right\} \tag{9}$$

where $\chi_{a,b} = L_{a,b}B_{a,b}$; $B_{a,b}$ is the function enclosed in square brackets in equation (4) and large brackets in equation (8) respectively.

To determine the differential probability $d\sigma_{a,b}(u)$, it is necessary to integrate (9) with respect to v, the splitting of the pair energy. It is most practicable to perform the integration numerically with respect to v. A difficulty arises in assigning an exact upper limit to v in view of the unspecific form of the relativistic restrictions previously mentioned. As it is not possible to state categorically the precise energy of a particle at which it may be regarded as relativistic, the usual procedure adopted is that introduced by Roe (1959) in which the range of v is defined as

$$|v| \leqslant 1 - \frac{nm}{uE}, \qquad n > 1 \tag{10}$$

with the choice of n unspecified. The value of n must be chosen so that (10) does not violate the relativistic requirements imposed by (2). A value of $n \simeq 4$ appears to satisfy these requirements and at the same time the range of v is not significantly reduced by this choice. Roe (1959) and Stoker *et al* (1963), for instance, have used lower values of n which, though not completely unjustified, can lead to negative contributions to the cross section. Such contributions have no physical significance and indicate a breakdown in the theory. However, present numerical computations have shown that, where secondary energies greater than 0·1 GeV are concerned, the actual assumed value of n is not critical, particularly for E' large. The restrictions imposed by (11) limit the range of u to

$$\frac{2nm}{E} \leqslant u < 1 - \frac{\mu}{E}. \tag{11}$$

The adjustable parameter α introduced by MUT in order to simplify the integration over angles, is known to be of the order of unity. The indeterminancy associated with this constant has a most marked effect on the computed cross section whenever E or E' is small. This parameter has been arbitrarily assigned values lying between 1 and 3 by, amongst others, Roe (1959, 1971), Stoker *et al* (1963), Allkofer *et al* (1971) and Bem *et al* (1969). In most instances the value of α is chosen to give best agreement between theory and experiment and appears most frequently as $\alpha = 2$.

3. The MUT approximate formulae

In an attempt to simplify (4), MUT have introduced two energy transfer domains. Domain I concerns small energy transfers for which $u \ll m/\mu$ while domain II refers to large energy transfers, $u \gg m/\mu$. The division of the (E, u) plane into domains is shown in figure 2, where additional subdivisions show the regions where screening is effective and the nonrelativistic regions where the formulae are not applicable. Similar diagrams are to be found in Kokoulin and Petrukhin (1970). Domain II is itself subdivided by MUT according to the value of v; $v^2 \ll 1$ refers to IIa and $1 - v^2 \ll 1$ to IIb (the meaning of a and b here is completely different from that in figure 1). This subdivision is a computational convenience only and cannot be shown in figure 2. The formulae applicable to the various regions are given by MUT in equations (30)–(34b). As these expressions appear to be in error, and since they are widely used, rederivation was considered worthwhile.

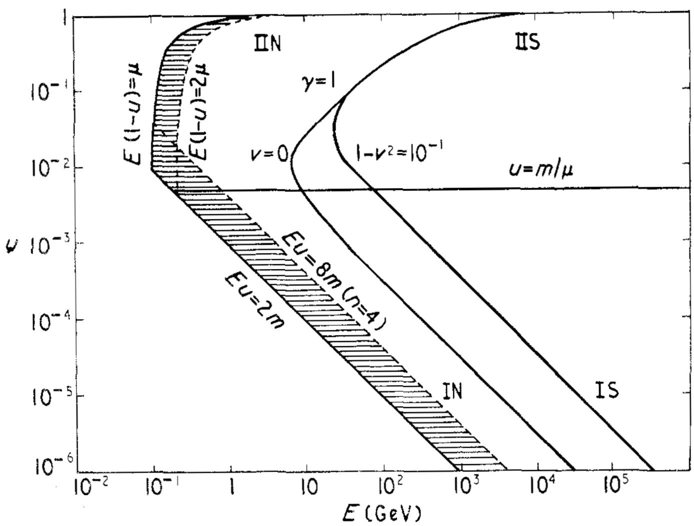

Figure 2. Regions available to pair production for $v = 0$ and $1 - v^2 = 10^{-1}$. The thick solid lines represent the physical limits for the process. Energy transfers in the shaded region are possible, but the formulae are not applicable here. The curves $\gamma = 1$ divide the plane into regions where screening is effective or absent.

For the case $x \ll 1$, we have from (4) for I and II

$$d^2\sigma_a = \frac{4}{3\pi}(Z\alpha' r_0)^2 \frac{du\, dv}{u} L_a \left\{ \left(1 - u + \frac{u^2}{2}\right)\left(1 + \frac{v^2}{2}\right) \ln\frac{1}{x} - (1-u)(1+v^2) - \frac{u^2}{4} \right\} \tag{12}$$

with

$$L_a = \ln(\alpha 137 Z^{-1/3}), \qquad \gamma \leqslant 1$$
$$= \ln\left(\frac{\alpha u E(1-v^2)}{2me}\right), \qquad \gamma > 1$$

In domain I, where $u \ll m/\mu$, $(1 - u + \frac{1}{2}u^2)$ and $(1-u)$ may be put equal to unity and $u^2/4$

may be neglected. Since $x \ll 1$ in domain I we have from (12)

$$d^2\sigma_{aI} = \frac{4}{3\pi}(Z\alpha' r_0)^2 \frac{du\, dv}{u} L_a \left\{ \left(1+\frac{v^2}{2}\right) \ln\frac{1}{x} - (1+v^2) \right\}. \tag{13}$$

It is interesting to note that equation (32) of Bhabha (1935) (no screening) is similar in form to equation (13) above. Expressing Bhabha's equation in terms of u and v we have

$$d^2\sigma(u, v) = \frac{4}{3\pi}(Z\alpha' r_0)^2 \frac{du\, dv}{u} \left(1+\frac{v^2}{2}\right) \ln\left(\frac{kuE(1-v^2)}{4m}\right) \ln\left(\frac{k'}{u}\frac{m}{\mu}\right). \tag{14}$$

In domain IIa, where $x \gg 1$ and $v^2 \ll 1$, (4) gives

$$\begin{aligned} d^2\sigma_{aIIa} &= \frac{4}{3\pi}(Z\alpha' r_0)^2 \frac{du\, dv}{u^3} L_a \left(\frac{m}{\mu}\right)^2 \frac{1-u}{1-v^2} \{(1+v^2)u^2 + (1-u)(3-v^2)\} \\ L_a &= \ln\left(\frac{\alpha 137 Z^{-1/3} \mu u}{2m(1-u)^{1/2}}\right), \qquad \gamma \leqslant 1 \\ &= \ln\left(\frac{\alpha E(1-u)^{1/2}}{\mu e}\right), \qquad \gamma > 1. \end{aligned} \tag{15}$$

The same functional dependence on u exhibited by Bhabha's cross section (discussed in the introduction) is apparent in equations (13) and (15).

In domain IIb, where $x \ll 1$ because $1 - v^2 \ll 1$, (12) is the appropriate formula. In the approximate formulae given by MUT (equations (30) and (34) in MUT), the expression for L_a is always taken as that pertaining to the case of no screening. It is clear from figure 2 that at very high energies this is not necessarily the most appropriate choice of L_a so that, in this respect alone, the MUT expressions will give misleading results. The choice of L_a should obviously be governed by the values of the parameter γ.

4. Other approximate expressions based on MUT

The approximate formulae (12), (13) and (15) are of little practical value; the computer time required for integration over v is of the same order as that for equation (4). In addition it becomes necessary to introduce yet another subjective criterion to govern the transition from region IIa to IIb.

A more satisfactory approximate formula than the above is that of Stoker and Haarhoff (1960) in which advantage is taken of the fact that for E and E' small the energy partition of the pair tends to be symmetric, that is, $\epsilon_+ \simeq \epsilon_-$. The variation of the partial cross section with v is accounted for to a sufficiently high degree of accuracy by making use of an empirical correction factor $C(u, E)$. This allows the cross section to be expressed in the following form, independent of v

$$d\sigma(u, E) = d\sigma(u)_{v=0} C(u, E). \tag{16}$$

It is clear from the computations in a later paper, Stoker *et al* (1963), that the formula is a reliable approximation even at higher energies of the order of 100 GeV.

Kobayakawa (1967), using the results of Ueda (private communication to Kobayakawa), has introduced a semi-empirical v independent formula of the following form

$$\Phi(E, u) = b_p(E)\phi(u)$$

with

$$\phi(u) = \frac{\delta_2(1+\delta_2)}{u(u+\delta_2)^2}, \qquad \delta_2 = \text{constant} \tag{17}$$

where $b_p(E)$ is the familiar energy loss expression for muons and takes into account the E dependence of the cross section, whereas $\phi(u)$ expresses the variation of cross section with u. Since formula (17) was originally introduced for the purpose of computing the mean energy loss $\mathrm{d}E/\mathrm{d}x$, discussion on its shortcomings is reserved for § 10.

5. The formula of Ternovskii

Ternovskii (1960a) has obtained an expression for the second order processes which does not contain the indeterminate constant α in the screening term. This is presumably a result of carrying out the integration over angles more accurately than earlier authors. By obtaining the identical expression for B_a as in MUT and the following for L_a

$$\begin{aligned} L_a &= \ln\left(\frac{uE(1-v^2)}{4m(1+x)^{1/2}}\right), \qquad \gamma > 1 \\ &= \ln\{137Z^{-1/3}(1+x)^{1/2}\}, \qquad \gamma \leqslant 1 \end{aligned} \tag{18}$$

it is clear by comparing (18) and (7) that α in MUT theory should be nearer 1 than 2. Ternovskii (1960b) gives consideration to the influence of multiple scattering (Migdal effect), but it would appear that this is relatively unimportant for primary energies less than 10^5 GeV. As in the case of electron induced bremsstrahlung and γ produced pairs, there is a suppression of the cross section at small energy transfers (Dovzhenko and Pomanskii 1965). A formula for first order processes is given by Ternovskii (equation (30) in Ternovskii 1960a), but the interpretation of this formula, which appears to be an approximation, is not clear.

6. The method of Zapolsky

In the thesis of Zapolsky (1962) cross sections under conditions of no screening and complete screening are derived. The main emphasis of the work is concentrated on pair production by incident electrons, and consequently cross sections for region I ($u < m/\mu$) are more carefully examined. (For electron primaries $m \equiv \mu$ and therefore $x \ll 1$ for most u of interest.) Equations (4.19) and (4.43) from the thesis, functions of u only, refer to the two cases of no screening and complete screening respectively. The cross section in domain II is represented by a function of constant logarithmic slope joining smoothly to $\mathrm{d}\sigma_{\mathrm{I}}$ at the upper edge of the domain

$$\mathrm{d}\sigma_{\mathrm{II}} = A\left(\frac{\mu}{2m}\frac{u}{1-u}\right)^{-p}, \qquad A = \text{constant} \tag{19}$$

where

$$p \equiv \frac{\mathrm{d}}{\mathrm{d}\ln u}[\ln \mathrm{d}\sigma(u)]_{u=m/\mu}$$

$$= -3\cdot1 - \frac{1\cdot07}{\ln E/\mu}.$$

7. The treatment of Kel'ner

According to the investigations of Kel'ner (1967), the previous theories are in agreement with one another except for energy transfers in region II. Kel'ner therefore derived exact expressions valid even for large energy transfers. These cross sections have a different form from those of the previous authors with the inclusion of terms independent of L_a. For second order processes, two expressions are given (equations (26) and (31) in Kel'ner 1967) which represent no screening and complete screening respectively.

Transforming these equations by means of (3) yields for Kel'ner's equation (26)

$$\mathrm{d}^2\sigma_a^{\mathrm{N}} = \frac{1}{3\pi}(Z\alpha' r_0)^2 \frac{(1-u)}{u}\,\mathrm{d}u\,\mathrm{d}v\left[\left\{2\ln\left(\frac{uE(1-v^2)}{2me^{1/2}(1+x)^{1/2}}\right)\right\}\left\{a_1 \ln\left(1+\frac{1}{x}\right) - b_1 - \frac{c_1}{1+x}\right\} - a_1 f\left(\frac{1}{1+x}\right) + b_1 x \ln\left(1+\frac{1}{x}\right) + \frac{c_1}{1+x}\right], \qquad \gamma > 1 \tag{20}$$

and for complete screening from (31)

$$\mathrm{d}^2\sigma_a^{\mathrm{S}} = \frac{1}{3\pi}(Z\alpha' r_0)^2 \frac{(1-u)}{u}\,\mathrm{d}u\,\mathrm{d}v\left[[2\ln\{183Z^{-1/3}(1+x)^{1/2}\}]\left\{a_1 \ln\left(1+\frac{1}{x}\right) - b_1 - \frac{c_1}{1+x}\right\} + a_1 f\left(\frac{1}{1+x}\right) - d_1 \ln\left(1+\frac{1}{x}\right) - \frac{2}{3}\frac{c_1}{1+x} + \frac{1}{6}(1-v^2)\right], \qquad \gamma \leqslant 1 \tag{21}$$

where

$$a_1 = (2+v^2)\left(1+\frac{u^2}{2(1-u)}\right) + x(3+v^2)$$

$$b_1 = (2+v^2)$$

$$c_1 = x + v^2 + \frac{u^2}{2(1-u)}$$

$$d_1 = b_1 x + \frac{(1-v^2)}{12}\left(\frac{1}{1-u} + (1-u)\right) + \frac{x}{6}(3-v^2)$$

$$f\left(\frac{1}{1+x}\right) = \int_0^{1/(1+x)} \frac{\ln(1-y)}{y}\,\mathrm{d}y = \sum_{n=1}^{\infty} \frac{1}{n^2}\left(\frac{1}{1+x}\right)^n.$$

These expressions contain terms that are not multiplied by the screening function L_a; Kokoulin and Petrukhin designate these terms by the function Δ (the last 3 terms in (20) and the last 4 in (21)). The Δ terms are not very different in (20) and (21), but are of opposite sign. Using the representation of (9) we have

$$B_a = 2\left\{a_1 \ln\left(1+\frac{1}{x}\right) - b_1 - \frac{c_1}{1+x}\right\}$$
$$= \left[2\left\{(2+v^2)\left(1+\frac{u^2}{2(1-u)}\right) + x(3+v^2)\right\} \ln\left(1+\frac{1}{x}\right)\right.$$
$$\left. -2(3+v^2) + \frac{2(1-v^2)}{1+x} - \frac{u^2}{(1-u)(1+x)}\right] \tag{22}$$

which bears a close resemblance, but is not identical, to B_a of MUT and Ternovskii.

The transformed expressions for the first order processes are given by Wright (1971) in formulae (4.30) and (4.31). Similar expressions for the first and second order processes are to be found in Rozental' (1968); these contain a few minor errors.

8. Kel'ner and Kotov (1968)

In a subsequent paper Kel'ner and Kotov (1968) use a different approach in which screening is accounted for in a continuous manner so that only one formula for second order processes is required instead of two, as previously. Their treatment of screening is essentially the same as that adopted by Petrukhin and Shestakov (1968) for bremsstrahlung. The cross section is not expressed in closed form, and a double integration over the four-momentum and v is required. The authors have computed cross sections for various incident and transferred energies, recording the results in tabular form.

9. Kokoulin and Petrukhin (1970, 1971)

A single analytical expression, including first and second order processes based on the work of Kel'ner (1967), has been given by these authors. The essential part of this formula is the introduction of a single, continuous function for $L_{a,b}$ which gives the identical expressions in the extreme cases of $\gamma \ll 1$ and $\gamma \gg 1$ to those of Kel'ner. For example, consider L_a only where

$$L_a = \ln\left(\frac{A'Z^{-1/3}(1+x)^{1/2}}{1+\gamma e^{1/2}}\right), \qquad A' = 183 \tag{23}$$

with

$$\gamma = \frac{2mA'(1+x)Z^{-1/3}}{Eu(1-v^2)}.$$

This function is identical to the screening term in (20) when $\gamma \gg 1$ and for $\gamma \ll 1$ is the same as the screening term in (21). As a first approximation (20) and (21) may be replaced by a single expression

$$\mathrm{d}^2\sigma_a = \frac{2}{3\pi}(Z\alpha' r_0)^2 \frac{(1-u)}{u}\, \mathrm{d}u\, \mathrm{d}v L_a B_a \tag{24}$$

where L_a and B_a are given by (23) and (22) respectively. The screening expression (23) is further modified to take account of the terms in (20) and (21) which are not functions of L_a (ie the Δ terms); the above treatment is repeated for the first order process, so that finally

$$\mathrm{d}^2\sigma_{a,b} = \frac{2}{3\pi}(Z\alpha' r_0)^2 \frac{(1-u)}{u}\left\{\phi_a + \left(\frac{m}{\mu}\right)^2 \phi_b\right\} \mathrm{d}u\,\mathrm{d}v \tag{25}$$

where ϕ_a and ϕ_b are given in their paper.

In the second paper the influence of the nuclear form factor on the cross section is taken into account. However, numerical computations show that the effects are extremely small: for $E > 10$ GeV and $u \leq 10^{-1}$ the cross section is reduced by less than 3%.

The authors verify the accuracy of the cross section represented in equation (25) by direct comparison with the Kel'ner and Kotov (1968) computations and find that agreement is within about 2%.

10. Numerical computation of the cross sections

All the expressions so far given for $\mathrm{d}^2\sigma_a$ have been computed for lead. In the case of the MUT cross sections, values for n of 4 and 10 were used, but for those of Kel'ner and Ternovskii only the case $n = 4$ was considered. In figure 3 the contributions to the differential probability from diagrams (a) and (b) of figure 1, as computed from equation (12) from Kokoulin and Petrukhin's paper, are shown. The insignificant contribution from digrams (b), except when $u \simeq 1$, is obvious.

To show the differences in the predicted cross sections as clearly as possible, according to the various formulae, they have been plotted relative to the cross section of Kokoulin and Petrukhin in figure (4). The effect of the parameter n is shown for the MUT formula;

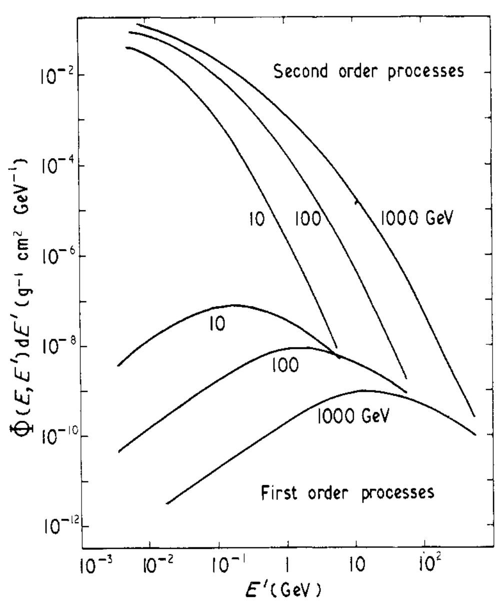

Figure 3. Differential energy transfer probabilities for lead, from the formula of Kokoulin and Petrukhin. Incident muon energies are indicated on the curves.

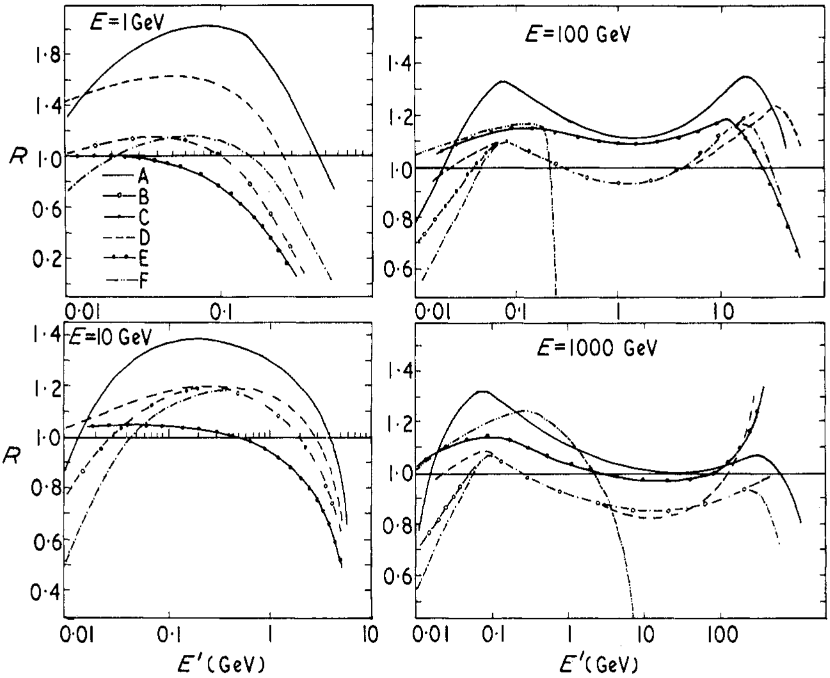

Figure 4. Ratio (R) of the differential probabilities, according to the formulae of various authors A MUT $\alpha = 2$, $n = 4$; B MUT $\alpha = 1$, $n = 4$; C MUT $\alpha = 1$, $n = 10$; D Ternovskii (1960a); E Kel'ner (1967); F Zapolsky (1962), relative to the predicted values of Kokoulin and Petrukhin.

for $E' < 0{\cdot}1$ GeV there is a reduction in the cross section as n is increased but little effect for $E' > 0{\cdot}1$ GeV, as would be expected. The choice of the parameter α for $E < 10$ GeV is noticeable over the whole range of energy transfers, although for $E > 100$ GeV the effect is much reduced. This behaviour is expected since α appears only in the logarithmic screening term. For $E = 1$ GeV the cross sections exhibit large differences but as the relativistic conditions imposed on the participating particles are more difficult to satisfy they cannot be regarded as accurate at this muon energy. At high energy transfers ($u \gtrsim 0{\cdot}3$) the predictions of the various formulae diverge strongly; it is concluded that all cross sections should be regarded as inaccurate in this region.

At energies of 10 and 1000 GeV the cross sections predicted by Kel'ner and Kotov (1968) agree to better than 2% with those of the Kokoulin and Petrukhin formula (these differences are too small to indicate on figure (4)). This was ascertained by using computed values of the function $F(E, u)$ as listed in table 1 of Kel'ner and Kotov (1968) for the muon energies of 10 and 1000 GeV. This agreement is, of course, an essential requirement, since the whole purpose of the method of Kokoulin and Petrukhin was to replace the complicated expression of Kel'ner and Kotov by something more readily computed.

The semi-empirical formula of Kobayakawa (1967), equation (17), predicts cross sections for rock which at $E \simeq 1000$ GeV are too low by 40% compared with the predictions of MUT equation (4). This can be directly ascribed to the fact that the Mando and Ronchi (1952) expression for $b_p(E)$ does not predict the same energy loss as the MUT cross section, contrary to what Kobayakawa claims. This formula is an inadmissible mixture of two theories and would be improved by retaining the function $\phi(u)$, with perhaps slight modification, together with the replacement of $b_p(E)$ by an expression based on the MUT theory.

From figure (4) we see that the cross sections of Zapolsky are in reasonable agreement with those of the other authors in region I but, in region II, the cross section is too low because (19) overestimates the slope—the cross sections of other authors, with the exception of Bhabha, exhibit a log slope of approximately -2 for $u = m/\mu$. Zapolsky's expressions (4.19) and (4.43) break down at the edge of domain I, and consequently equation (19) cannot be taken as reliable.

11. The rate of energy loss due to pair production

The rate of energy loss for a muon is given by

$$-\frac{\mathrm{d}E}{\mathrm{d}x} = \frac{EN}{A}\iint \mathrm{d}^2\sigma(u, v)u\,\mathrm{d}u\,\mathrm{d}v = b_p(E)E. \tag{26}$$

This double integral was numerically evaluated for standard rock ($Z = 11$, $A = 22$) using the theory of MUT in the two cases $\alpha = 1$ and 2 with $n = 10$. Account was taken of the contribution to the energy loss by atomic electrons by replacing Z^2 in the cross section by $Z(Z+a)$. From Kel'ner and Kotov (1968) for complete screening $a = 1{\cdot}3$ and, as the major contributions to the integral are made under conditions of complete screening (see figure 2), negligible error results from assuming $a = 1{\cdot}3$ for all u and v.

In figure 5 energy loss relationships computed from MUT theory are shown together with the calculations of Kel'ner and Kotov (1968), Mando and Ronchi (1952) and

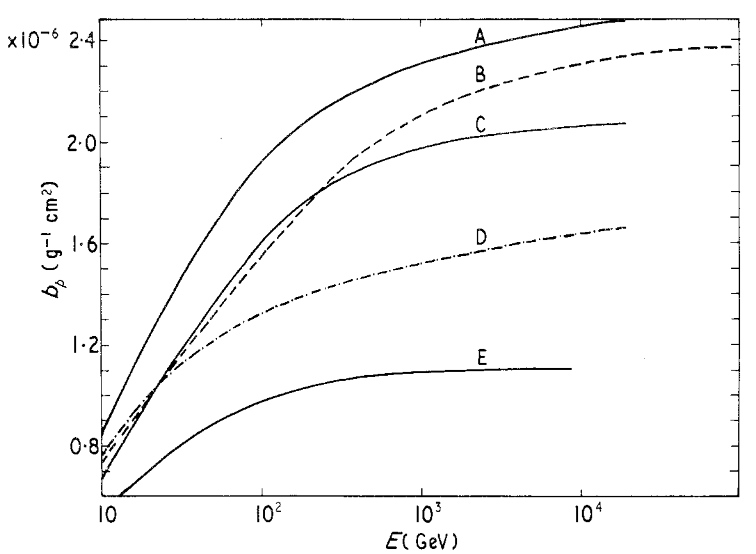

Figure 5. The pair production energy loss coefficient for standard rock. A MUT $\alpha = 2$; B Kel'ner and Kotov (1968); C MUT $\alpha = 1$; D Mando and Ronchi (1952); E Castagnoli *et al* (1964).

Castagnoli *et al* (1964). The extremely low rate of energy loss obtained by Castagnoli *et al* emphasizes the inadvisability of using the approximate MUT formulae. Separate calculations for the formula of Kokoulin and Petrukhin were not made simply because such calculations would give the same results as those of Kel'ner and Kotov.

Following Kobayakawa, it is convenient to express $b_p(E)$ in empirical form. For the curve of Kel'ner and Kotov we have

$$b_p(E) = \left(0{\cdot}37\ln\frac{E}{\mu} - 0{\cdot}95\right)\times 10^{-6}\ \mathrm{g^{-1}\,cm^2}, \qquad E < 100\,\mathrm{GeV}$$

$$= 2{\cdot}75\left(\frac{\ln E/\mu - 5{\cdot}43}{\ln E/\mu - 4{\cdot}34}\right)\times 10^{-6}, \qquad 100 \leqslant E \leqslant 5\times 10^{4}. \tag{27}$$

12. Conclusions

It would appear that the basic treatments of the process of pair production by MUT, Ternovskii, Kel'ner, Kel'ner and Kotov and Kokoulin and Petrukhin are all in reasonable agreement, particularly for high energy muons, $E > 100$ GeV. At low energy transfers the various formulae exhibit very different characteristics from one another; in most cases these differences are readily explained. It is clear that the cross sections are not meant to apply to energy transfers below 10 MeV and predictions in this region must be regarded with caution. It is not obvious why the expressions of Kel'ner and Kokoulin and Petrukhin exhibit such striking differences when $E' \simeq E$; according to Kel'ner, the formulae should be accurate in this region. Figure (2) illustrates that, for muons with energy > 10 GeV, most energy transfers take place under conditions of complete screening: this point appears not to have been fully appreciated in the past.

Clearly if numerical cross sections are required the most convenient expression to use is that of Kokoulin and Petrukhin; it has been firmly established that this formula does give the same results as that of Kel'ner and Kotov, while having the advantage of being easy to compute. With regard to approximate formulae it is best, now that high speed computers are readily available, to avoid them altogether, although, for the purpose of a quick check, the expression of Stoker *et al* (1963) is useful for hand calculation.

The earlier estimates of $b_p(E)$ of Mando and Ronchi (1952), Hayman *et al* (1963), Kobayakawa (1967) and Castagnoli *et al* (1964) are not in accord with those obtained by direct integration of the cross sections of MUT, Kel'ner and Kotov, Ternovskii or Kokoulin and Petrukhin. All the latter cross sections predict $b_p(E)$ values in the region of $2\text{–}2{\cdot}6\times 10^{-6}$, whereas in the former, values in the range $1{\cdot}1\text{–}1{\cdot}7\times 10^{-6}$ are given. This conclusion is supported by the work of Erlykin (1966), Rozental' (1968), Meyer *et al* (1970) and Kotov and Logunov (1970).

The various approaches adopted in the treatment of pair production are essentially variants of the Feynman method. Each author has paid more attention to the mathematical details of the derivations than has his predecessor. Apart from inaccuracies introduced in presenting the cross sections in closed form, there are various physical effects which have been neglected; amongst the most significant corrections discussed by Rozental' (1968) are those of the influence of orbital electrons and radiative corrections. He concludes that the magnitude of the uncertainty, arising from all the physical assumptions made, is of the order of 5%.

Acknowledgments

I wish to thank Drs J C Barton and I W Rogers for their constant encouragement in this work.

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